Journal of Economic Theory **107**, 356–376 (2002)
doi:10.1006/jeth.2001.2964

Intertemporal Equity and Efficient Allocation of Resources[1]

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Received November 28, 2000; final version revised August 13, 2002

Competitive paths which are efficient are shown to satisfy a terminal cost minimization condition, thereby providing a continuous-time counterpart to the discrete-time result due to Malinvaud. Using this result, competitive paths which are equitable and efficient are shown to satisfy Hartwick's investment rule, which states that the value of net investment is zero at each date. Our result indicates that Hartwick's rule can help to signal inefficiency of competitive equitable paths. *Journal of Economic Literature* Classification Numbers: C61, D90, O41.

Key Words: intertemporal equity; efficiency; Hartwick's rule; terminal cost minimization.

1. INTRODUCTION

The purpose of this paper is to study the properties of efficient equitable paths in an infinite-horizon continuous-time framework with heterogenous capital goods (which can include non-renewable resource stocks). It is shown that competitive paths which are efficient satisfy a "terminal cost minimization" condition in the sense of Malinvaud [18]. This is used to establish the principal result of the paper: competitive paths, which are both equitable and efficient, must satisfy the condition that the value of net investment must be equal to zero at each date.

We now relate these results to those available in the literature. This area of study originates with a paper by Solow [19], who analyzed a capital accumulation model, with Cobb–Douglas technology, in the presence of an exhaustible resource. He was interested in the possibility of sustainable

[1] I thank Swapan Dasgupta for many useful conversations. An earlier version of this paper was presented at a Conference on "General Equilibrium and Economic Growth" at Queen Mary and Westfield College, University of London, in June, 2000. Comments from participants at the conference are gratefully acknowledged. The current version has benefited from suggestions by Christian Ghiglino and a referee of this journal.

0022-0531/02 $35.00

consumption levels in this context and concentrated attention on growth paths which maximized the welfare of the least well off generation among all growth paths feasible from given initial resources.[2] Such "maximin" paths are efficient as well as equitable, where equity in this context means that the path maintains a constant consumption level at all dates.

Subsequently, Hartwick [14] made the interesting observation that a competitive equilibrium path, which follows the simple rule of thumb of investing the rents from the exhaustible resource used at each date,[3] in the net accumulation of the produced capital good, is equitable. As Solow [20, 21] has observed, this is an intuitively appealing investment rule of maintaining the consumption potential of society, in a generalized sense, by replacing exhaustible resource stocks, which are used up, with produced capital goods of equal value.

It turns out that Hartwick's observation has significance in a wider class of models than the special context in which it arose initially. In particular, Dixit *et al.* [12] recognized that Hartwick's investment rule is really a statement that the valuation of net investment (including the disinvestment in the exhaustible resource) is zero at each date.[4] They showed in a general model of accumulation involving heterogenous capital goods (which could include various non-renewable resource stocks) that if the valuation of net investment along a competitive path is *constant* over time[5] (the constant is not required to be zero) then this would ensure intertemporal equity (in the sense described above, but with "consumption" interpreted now as the utility based on a vector of consumption goods). Furthermore, this investment rule was shown to be a *necessary* condition for intertemporal equity along competitive paths.

This is a complete characterization of competitive equitable paths, and it naturally leads one to re-examine the significance of Hartwick's investment rule. This is prompted by the fact that in Solow's original exercise in the context of the exhaustible resource model, the maximin equitable paths do in fact satisfy Hartwick's rule, not just the Dixit–Hammond–Hoel rule.

This observation leads to the conjecture that for competitive paths which are both equitable and efficient, Hartwick's rule should hold. In the exhaustible resource model (but without the special structure of the Cobb–Douglas technology of Solow [19]), a result like this was first noted by Dasgupta and Mitra [8]. However, their treatment of equity and efficiency was in the context of a discrete-time model, where Hartwick's rule

[2] That is, the objective function in his exercise was of the Rawlsian Maximin type.

[3] This is often referred to, in the literature, as Hartwick's (investment) rule.

[4] In subsequent discussions, it is this more general formulation of the investment rule that will be referred to as Hartwick's rule (HR).

[5] This investment rule will be referred to subsequently as the Dixit–Hammond–Hoel rule (DHHR).

does not hold in the original form but rather in a modified form. There has been quite a bit of interest in this issue more recently. In the continuous time framework of this exhaustible resource model, Hartwick's rule does hold in its original form as a necessary condition along efficient equitable paths (see Withagen and Asheim [24] for references to some of the literature that has emerged).

Given these results in the exhaustible resource model, Withagen and Asheim [24] posed the following general problem: "A question that naturally arises is whether the converse of Hartwick's rule holds in general in an economy with stationary instantaneous preferences and a stationary technology: Does an efficient constant utility path imply that the value of net investments equals zero at each point in time?"[6] They answered the question in the affirmative for efficient constant utility paths that are supported by positive utility discount factors having the property that the integral of the discount factors exists; that is, for paths which are *regular maximin* in the sense of Burmeister and Hammond [1]. The difficulty is that with this restriction they cannot accomodate into their theory paths of the golden-rule variety, in neoclassical models of the Cass–Koopmans type, in which exhaustible resources are not essential factors in production.

In order to answer the question stated above, without any further restrictions in its scope, one needs a suitable *necessary* condition for efficiency in a continuous-time framework.[7] The necessary condition of efficiency that is most useful in the present context is the "terminal cost minimization" condition of Malinvaud [18], because it provides precisely the information required to show the necessity of Hartwick's rule for efficient equitable paths. In fact, Hartwick's rule is seen to be the necessary first-order condition of terminal cost minimization along competitive equitable paths.

The paper is organized as follows. In Section 2, we present a general model of intertemporal allocation in the continuous time framework, along the lines of Cass and Shell [5]. In Section 3, we discuss Hartwick's rule and its generalization by Dixit *et al.* [12]. We then discuss these rules in the context of the standard one-sector neoclassical model and provide an example where a competitive path which is equitable does *not* satisfy

[6] Dixit *et al.* [12] attempted to answer this question, but they were only able to establish this under a "capital deepening" condition used by Burmeister and Turnovsky [2], which is not easy to interpret.

[7] Unfortunately, the study of efficient allocation of resources has been confined, almost exclusively, to the discrete-time framework. The important characterizations of efficiency, due to Malinvaud [18] and Cass [4], are established in discrete-time models. While there are some characterizations of efficiency in the continuous-time framework (see, for example, Majumdar [17], for some of the important theorems), general results exclude the settings which allow for golden-rule type programs.

Hartwick's rule. In Section 4, we present the main results of the paper: (i) competitive paths which are efficient are shown to satisfy the "terminal cost minimization condition," and (ii) competitive paths which are equitable and efficient are shown to satisfy Hartwick's rule.

Given our results, Hartwick's rule takes on a significance that is quite distinct from the intertemporal equity issue. In fact, its role is seen to be to help signal *inefficient* equitable paths. Any competitive equitable path would satisfy the Dixit–Hammond–Hoel rule; but, if the (constant) value of net investment were non-zero (that is, if Hartwick's rule were violated), it would be necesarily inefficient.

2. PRELIMINARIES

2.1. *The Framework*

Consider a framework in which population and technology are unchanging, individuals at each date are identical in all respects (so one can think in terms of a single representative person at each date and ignore distribution considerations).

Denote by $k_i \geqslant 0$, the stock of the ith capital good, where $i = 1, ..., n$, and by z_i the investment flow, net of depreciation, of the ith capital good. Denote the vectors $(k_1, ..., k_n)$ and $(z_1, ..., z_n)$ by k and z respectively. The *technology set*, denoted by Λ, is a set of pairs (z, k) in $\mathbb{R}^n \times \mathbb{R}^n_+$. By a typical point (z, k) of Λ we understand that from capital input stock k it is technologically feasible to obtain the flow of net investment z. The (instantaneous) *welfare function* is denoted by a function $u: \Lambda \to \mathbb{R}$. We shall make the following assumptions[8] on Λ and u.

(A.1) Λ is closed and convex; for each $k \geqslant 0$, there is a z in $\mathbb{R}^n$, such that $(z, k) \in \Lambda$.

(A.2) Given any number $\zeta > 0$ there is a number $\eta > 0$ such that $(z, k) \in \Lambda$ and $|k| \leqslant \zeta$ implies $|u(z, k)| \leqslant \eta$ and $|z| \leqslant \eta$.

(A.3) u is continuous on Λ and twice continuously differentiable in the interior of Λ.

(A.4) $u(z, k) \geqslant 0$ for $(z, k) \in \Lambda$; $u(z, k) \geqslant u(z', k)$ if (z, k) and $(z', k) \in \Lambda$ and $z \leqslant z'$.

(A.5) u is a concave function on Λ; for each $k \in \mathbb{R}^n_{++}$, $u(z, k)$ is a strictly concave function of z; that is, if (z', k) and (z, k) are in Λ satisfying

[8] For x, y in $\mathbb{R}^n$, $x \geqslant y$ means $x_i \geqslant y_i$ for $i = 1, ..., n$; $x > y$ means $x \geqslant y$ and $x \neq y$; $x \gg y$ means $x_i > y_i$ for $i = 1, ..., n$. For x in $\mathbb{R}^n$, the sum norm of x, denoted by $\|x\|$ is defined by $\|x\| = \sum_{i=1}^n |x_i|$.

$z' \neq z$ and λ is a number satisfying $0 < \lambda < 1$, then $u(\lambda z + (1-\lambda) z', k) > \lambda u(z, k) + (1-\lambda) u(z', k)$; in the interior of Λ, the matrix of second partials of u with respect to z, $[\partial^2 u(z, k)/\partial z^2]$, is negative definite.

For each $k \geqslant 0$, defining the set $\Lambda(k)$ by: $\Lambda(k) \equiv \{z: (z, k) \in \Lambda\}$, we note that $\Lambda(k)$ is a non-empty, compact, and convex subset of $\mathbb{R}^n$.

A *path* from initial stock K in $\mathbb{R}^n_+$ is a pair of functions $(z(\cdot), k(\cdot))$, where $z(\cdot): [0, \infty) \to \mathbb{R}^n$ and $k(\cdot): [0, \infty) \to \mathbb{R}^n_+$, such that $k(\cdot)$ is absolutely continuous and[9]

$$\begin{aligned} &(z(t), k(t)) \in \Lambda \quad \text{for } t \geqslant 0, \text{ a.e.;} \\ &\dot{k}(t) = z(t) \quad \text{for } t \geqslant 0, \text{ a.e.;} \quad \text{and} \quad k(0) = K. \end{aligned} \tag{2.1}$$

Denote by $\mathfrak{I}(K)$ the set of paths from initial stock K. We shall assume

(A.6) For each K in $\mathbb{R}^n_+$, $\mathfrak{I}(K)$ is non-empty.

A path $(z(t), k(t))$ from K is called *equitable* if $u(z(t), k(t))$ is constant over time. It is called *inefficient* if there is another path $(z'(t), k'(t))$ from K, such that $u(z'(t), k'(t)) \geqslant u(z(t), k(t))$ for $t \geqslant 0$, a.e., and denoting Lebesgue measure on the reals by μ,

$$\mu\{t: u(z'(t), k'(t)) > u(z(t), k(t))\} > 0. \tag{2.2}$$

It is called *efficient* if it is not inefficient.

2.2. *Examples*

In this section we shall provide two examples of the framework described earlier. The examples will play a role in subsequent sections. They have not been chosen for their generality; various multisector models which may be accomodated in the framework described earlier may be found in the examples discussed in Magill [16].

EXAMPLE 1. This is the well known one sector neoclassical growth model of the Cass–Koopmans type (see Cass [3], Koopmans [15]).

[9] We are using conventional notation: $\dot{x}$ means the time derivative of x. So if $x(t) = (x_1(t), ..., x_n(t))$, then $\dot{x}(t) = [dx_1/dt, ..., dx_n/dt]$. If x is a vector $(x_1, ..., x_n)$ and $f(x)$ is a vector valued function defined from $\mathbb{R}^n$ to $\mathbb{R}^m$, that is, $f(x) = ((f_1(x), ..., f_m(x))$, then $f'(x)$ is the $m \times n$ matrix whose ijth element is $(\partial f_i(x)/\partial x_j)$. If $f: \mathbb{R}^n \to \mathbb{R}$, then f_i is the ith partial derivative of f and f_{ij} is the jth partial derivative of f_i, for $i = 1, ..., n$, $j = 1, ..., n$. The notation "a.e" stands for "almost everywhere"; more precisely, if A is a subset of $\mathbb{R}$, then by the expression "for $t \in A$, a.e." we mean "for $t \in B$, where B is a subset of A such that the complement of B in A is a set of Lebesgue measure zero"; if the set A is an interval $[a, \infty)$ we often use the expression "for $t \geqslant a$, a.e." in place of "for $t \in [a, \infty)$, a.e."

There is one good which is both the capital good and the consumption good. Labor is assumed to be constant over time. Let $G\colon \mathbb{R}_+ \to \mathbb{R}_+$ denote the gross production function; a number δ, satisfying $0 < \delta < \infty$, denotes the constant exponential rate of depreciation of the capital stock; and $w\colon \mathbb{R}_+ \to \mathbb{R}_+$ denotes a cardinal welfare function. The functions G and w are assumed to satisfy the following properties:

(N.1) $G(0) = 0$; G is continuous on $\mathbb{R}_+$; G is twice continuously differentiable on $\mathbb{R}_{++}$; for $k > 0$, $G'(k) > 0$ and $G''(k) \leqslant 0$; there is $k' > 0$ such that for $k \in (0, k']$, $G'(k) > \delta$; there is $k'' > 0$ such that for $k \in [k'', \infty)$, $G'(k) < \delta$.

(N.2) $w(0) = 0$; w is continuous and concave on $\mathbb{R}_+$; w is twice continuously differentiable on $\mathbb{R}_{++}$; $w'(C) > 0$ and $w''(C) < 0$ for all $C > 0$; $w'(C) \to \infty$ as $C \to 0$.

The technology set here is $\Lambda = \{(z, k)\colon k \geqslant 0;\ G(k) - \delta k \geqslant z \geqslant -\delta k\}$; $u\colon \Lambda \to \mathbb{R}_+$ is given by the formula: $u(z, k) = w(G(k) - \delta k - z)$, for $(z, k) \in \Lambda$. It may be verified that Example 1 satisfies (A.1) to (A.6). Details may be found in Dasgupta and Mitra [9].

EXAMPLE 2. This is a standard model employed in the literature on optimal allocation of resources over time in the presence of an exhaustible resource (see for example Dasgupta and Heal [6, 7], Solow [19]).

In this model, there is one produced good, which serves as both the capital and the consumption good, and there is an exhaustible resource. Labor is assumed to be constant over time. Denote by k_1 the stock of augmentable capital good and by k_2 the stock of the exhaustible resource. A number δ, satisfying $0 \leqslant \delta < \infty$, denotes the constant exponential depreciation rate of augmentable capital. Let $G\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ denote the gross production function for the capital cum consumption good, using the capital input stock k_1 and the flow of exhaustible resource used $(-z_2)$. It is assumed that the flow of resource use cannot exceed a maximum level denoted by $R > 0$. The output $G(k_1, -z_2)$ can be used to replace worn out capital (if any), δk_1, to augment the capital stock through net investment, z_1, or to provide consumption, C, which generates utility according to a (welfare) function $w\colon \mathbb{R}_+ \to \mathbb{R}_+$.

The following assumptions are made on G and w:

(R.1)

(i) $G(0, 0) = G(0, y) = G(x, 0) = 0$ for $x \geqslant 0$ and $y \geqslant 0$.

(ii) G is continuous, concave and nondecreasing on $\mathbb{R}^2_+$, and twice continuously differentiable on $\mathbb{R}^2_{++}$; further, $G_1(x, y) > 0$, $G_2(x, y) > 0$, and $G_{11}(x, y) < 0$, $G_{22}(x, y) < 0$ for $(x, y) \gg 0$.

(R.2) $w(0)=0$, w is continuous and concave on $\mathbb{R}_+$; w is twice continuously differentiable on $\mathbb{R}_{++}$; $w'(C)>0$ and $w''(C)<0$ for $C>0$; $w'(C)\to\infty$ as $C\to 0$.

The technology set here is $\Lambda=\{(z_1, z_2, k_1, k_2)\colon (k_1, k_2)\geqslant 0;\ -R\leqslant z_2\leqslant 0;\ G(k_1, -z_2)-\delta k_1\geqslant z_1\geqslant -\delta k_1\}$ and the formula for $u\colon \Lambda\to\mathbb{R}_+$ is: $u(z_1, z_2, k_1, k_2)=w(G(k_1, -z_2)-\delta k_1-z_1)$ for $(z_1, z_2, k_1, k_2)\in\Lambda$. It may also be verified that Example 2 satisfies (A.1) to (A.6). Details may be found in Dasgupta and Mitra [9].

2.3. *Competitive Paths*

We shall now elaborate what we mean by a time path of quantities and prices which evolve along an equilibrium of a competitive market economy, from an initial stock K. It would be convenient, for what follows, to introduce the following notation and concepts. Let $p=(p_1, ..., p_n)$ denote prices of the investment goods and q denote a positive weight on the utility. Define a function $H\colon \mathbb{R}^n_+\times\mathbb{R}^n\times\mathbb{R}_{++}\to\mathbb{R}$ by

$$\left.\begin{aligned} H(k, p, q) &= \text{Maximize } [qu(z, k)+pz] \\ &\quad \text{subject to } (z, k)\in\Lambda \end{aligned}\right\} \tag{H}$$

For each k in $\mathbb{R}^n_+$, $\Lambda(k)$ is non-empty and compact and so $H(k, p, q)$ is well defined. Further, H is convex in p and q, and since Λ is convex, H is concave in k.

By (A.5), for $k\in\mathbb{R}^n_{++}$, $u(z, k)$ is strictly concave in z and, therefore, there is a unique maximizing choice of investment, which solves (H). We can write this maximizing choice of z in (H) as a function $g(k, p, q)$; that is, $g\colon \mathbb{R}^n_+\times\mathbb{R}^n\times\mathbb{R}_{++}\to\mathbb{R}^n$ is such that $(g(k, p, q), k)\in\Lambda$, and

$$H(k, p, q)=qu(g(k, p, q), k)+pg(k, p, q).$$

Let (k^0, p^0, q^0) satisfy $k^0\in\mathbb{R}^n_{++}$, and $(g(k^0, p^0, q^0), k^0)\in \textit{int}\,\Lambda$. Then, by the definition of g, the first-order condition of problem (H) yields

$$p^0+q^0\,\partial u(g(k^0, p^0, q^0), k^0)/\partial z=0.$$

By (A.3), the function $f(k, p, q, z)\equiv p+q[\partial u(z, k)/\partial z]$ is defined in an open neighborhood around $(k^0, p^0, q^0, g(k^0, p^0, q^0))$, it is continuously differentiable, and its derivative matrix with respect to z, evaluated at $(k^0, p^0, q^0, g(k^0, p^0, q^0))$, is non-singular. Thus, by the implicit function theorem, $g(k, p, q)$ is continuously differentiable with respect to (k, p, q) in an open neighborhood N of (k^0, p^0, q^0) and the range of $(g(k, p, q), k)$, for (k, p, q) in N, is in an open subset of Λ. It follows that in this neighborhood N of (k^0, p^0, q^0), H is continuously differentiable and, by the envelope theorem,

$$\partial H(k, p, q)/\partial p = g(k, p, q); \qquad \partial H(k, p, q)/\partial q = u(g(k, p, q), k) \tag{2.3}$$

$$\partial H(k, p, q)/\partial k = \partial u(g(k, p, q), k)/\partial k. \tag{2.4}$$

A *competitive path* is a path $(z(t), k(t))$ with associated prices, denoted by absolutely continuous functions of time $q(t)$ and $(p_1(t), ..., p_n(t)) \equiv (p(t))$, with $q(t) > 0$ and $p(t) \geqslant 0$ for $t \geqslant 0$, a.e., satisfying the following two conditions:

$$q(t)\, u(z(t), k(t)) + p(t)\, z(t) = H(k(t), p(t), q(t)) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{2.5}$$

$$\dot{p}(t) = -\partial H(k(t), p(t), q(t))/\partial k \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{2.6}$$

Here, $p(t)$ is the vector of present value prices of the investment goods, prevailing along a competitive path, at date t. Use the notation $(z(t), k(t), p(t), q(t))$ to denote a competitive path with its associated prices. Along a competitive path, for each $t \geqslant 0$, we denote $H(k(t), p(t), q(t))$ by $y(t)$; that is,

$$y(t) \equiv H(k(t), p(t), q(t)) \qquad \text{for} \quad t \geqslant 0. \tag{2.7}$$

Interpreting utility as an output with present value price $q(t)$, (2.5) says that the maximum value of output achievable from capital stocks $k(t)$ at the prices $p(t)$, $q(t)$ [that is, $H(k(t), p(t), q(t))$] is realized along a competitive path:

$$y(t) = q(t)\, u(z(t), k(t)) + p(t)\, z(t) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{2.8}$$

Equation (2.6) says that asset markets are in equilibrium; that is, no gains can be made by pure arbitrage (see Dorfman *et al.* [13], Weitzman [23], for expositions of this no-arbitrage principle).

If $(z(\cdot), k(\cdot))$ is a path from K in $\mathbb{R}^n_+$, we shall say that it is *interior* if (i) $(z(t), k(t))$ is in the interior of Λ in $\mathbb{R}^n \times \mathbb{R}^n$ for $t \geqslant 0$, a.e., and (ii) $k(t) \in \mathbb{R}^n_{++}$ for $t \geqslant 0$. We now note a preliminary result for interior competitive paths, which will be used in the next section.

LEMMA 1. *If $(z(t), k(t), p(t), q(t))$ is an interior competitive path from K in $\mathbb{R}^n_{++}$, then*

(i) *the function* $y(t)$, *defined in* (2.7), *is an absolutely continuous function of* t; *and*

(ii) $\dot{y}(t) = \dot{q}(t)\, u(z(t), k(t))$ *for* $t \geqslant 0$, *a.e.*

Proof. (i)[10] Let $0 \leqslant a < b < \infty$ be given. For $t \in [a, b]$, we have $y(t) = H(k(t), p(t), q(t))$. Now $(k(t), p(t), q(t))$ are continuous on $[a, b]$, so we can find $0 \leqslant m \leqslant M < \infty$ such that for all $t \in [a, b]$, $m \leqslant k_i(t) \leqslant M$ for $i = 1, ..., n$, $0 \leqslant p_i(t) \leqslant M$ for $i = 1, ..., n$, and $m \leqslant q(t) \leqslant M$. Since the competitive path is interior, we may choose $m > 0$. Thus, $E \equiv [me, Me]$ is a compact subset in the interior of $\mathbb{R}^n_+$, where $e = (1, ..., 1)$ in $\mathbb{R}^n$.

Let t_1, t_2 be arbitrary points in $[a, b]$. Then $y(t_2) - y(t_1) = H(k(t_2), p(t_2), q(t_2)) - H(k(t_1), p(t_1), q(t_1)) = H(k(t_2), p(t_2), q(t_2)) - H(k(t_1), p(t_2), q(t_2)) + H(k(t_1), p(t_2), q(t_2)) - H(k(t_1), p(t_1), q(t_1))$. Thus, we have

$$|y(t_2) - y(t_1)| \leqslant |H(k(t_2), p(t_2), q(t_2)) - H(k(t_1), p(t_2), q(t_2))| + |H(k(t_1), p(t_2), q(t_2)) - H(k(t_1), p(t_1), q(t_1))|. \quad (2.9)$$

The function $h(k) = H(k, p(t_2), q(t_2))$ is a concave function on $\mathbb{R}^n_+$ and, therefore, is Lipschitz on the compact subset E in the interior of $\mathbb{R}^n_+$, with Lipschitz constant $L_1 > 0$. Thus, we have

$$|H(k(t_2), p(t_2), q(t_2)) - H(k(t_1), p(t_2), q(t_2))| \leqslant L_1\, |k(t_2) - k(t_1)|. \quad (2.10)$$

The function $g(p, q) = H(k(t_1), p, q)$ is a convex function on $\mathbb{R}^n \times \mathbb{R}_{++}$, and therefore, is Lipschitz on the compact set $[0, Me] \times [m, M]$ in the interior of $\mathbb{R}^n \times \mathbb{R}_{++}$, with Lipschitz constant $L_2 > 0$. Thus, we have

$$|H(k(t_1), p(t_2), q(t_2)) - H(k(t_1), p(t_1), q(t_1))| \leqslant L_2\, |(p(t_2), q(t_2)) - (p(t_1), q(t_1))|. \quad (2.11)$$

Given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $a_1, b_1, ..., a_r, b_r$ are numbers satisfying $a \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_r < b_r \leqslant b$ and $\sum_{j=1}^{r} (b_j - a_j) < \delta$, then for all $i = 1, ..., n$, $\sum_{j=1}^{r} |k_i(b_j) - k_i(a_j)| < (\varepsilon/3nL_1)$, and for all $i = 1, ..., n$, $\sum_{j=1}^{r} |p_i(b_j) - p_i(a_j)| < (\varepsilon/3nL_2)$, and $|q(b_j) - q(a_j)| < (\varepsilon/3L_2)$, since k, p and q are absolutely continuous on $[a, b]$. Thus, using (2.9), (2.10), and (2.11), we have

$$\sum_{j=1}^{r} |y(b_j) - y(a_j)| \leqslant L_1(\varepsilon/3L_1) + L_2(\varepsilon/3L_2) + L_2(\varepsilon/3L_2) = \varepsilon,$$

which means that $y(t)$ is absolutely continuous on $[a, b]$.

[10] This proof is a slight modification of the proof used in Dasgupta and Mitra [10, p. 433].

(ii) Since $g(k, p, q)$ solves the problem (H) for $k \in \mathbb{R}^n_{++}$, we can use condition (2.5) for a competitive path to obtain

$$g(k(t), p(t), q(t)) = z(t) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{2.12}$$

Since the path is interior, $(g(k(t), p(t), q(t)), k(t))$ is in the interior of Λ. By (i) above, $y(t)$ is absolutely continuous, and so for $t \geqslant 0$ a.e., $y(t)$ is differentiable. Also, $k(t)$, $p(t)$ and $q(t)$ are differentiable, and $H(k, p, q)$ is continuously differentiable at $(k(t), p(t), q(t))$. Thus, by differentiating (2.7), and using (2.3), we get

$$\begin{aligned}\dot{y}(t) &= [\partial H(k(t), p(t), q(t))/\partial k]\, \dot{k}(t) + [\partial H(k(t), p(t), q(t))/\partial p]\, \dot{p}(t) \\ &\quad + [\partial H(k(t), p(t), q(t))/\partial q]\, \dot{q}(t) \\ &= [\partial H(k(t), p(t), q(t))/\partial k]\, \dot{k}(t) + g(k(t), p(t), q(t))\, \dot{p}(t) \\ &\quad + \dot{q}(t)\, u(z(t), k(t)) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.}\end{aligned}$$

Thus, we obtain

$$\dot{y}(t) = [\partial H(k(t), p(t), q(t))/\partial k]\, \dot{k}(t) + \dot{k}(t)\, \dot{p}(t) + \dot{q}(t)\, u(z(t), k(t)) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{2.13}$$

Combining condition (2.6) for a competitive path with (2.13), we have

$$\dot{y}(t) = \dot{q}(t)\, u(z(t), k(t)) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{2.14}$$

which establishes (ii). ∎

Remark 1. If $(q(t))$ is an absolutely continuous function, with $q(t) > 0$ for $t \geqslant 0$, and $(z(t), k(t))$ is a path from K, which is *optimal* in the sense that

$$\infty > \int_0^\infty q(t)\, u(z(t), k(t))\, dt \geqslant \int_0^\infty q(t)\, u(z'(t), k'(t))\, dt$$

for all paths $(z'(t), k'(t))$ from K, then one can use the Main Theorem of Takekuma [22, p. 431] to obtain an absolutely continuous function $(p(t))$, with $p(t) \in \mathbb{R}^n_+$ for $t \geqslant 0$, such that $(z(t), k(t), p(t), q(t))$ is a competitive path from K; that is, $(z(t), k(t), p(t), q(t))$ satisfies (2.5) and (2.6). (In fact, Takekuma's result can be applied to get this result in the more general case, where $(z(t), k(t))$ is a path from K, which is optimal in the sense that it is not overtaken by any path $(z'(t), k'(t))$ from K.) Interpreting $q(t)$ as the variable discount factor applied to the utility obtained at time t, we see that our analysis of the behavior of competitive paths includes the behavior of the optimal paths considered by Withagen and Asheim [24].

3. A CHARACTERIZATION OF EQUITABLE PATHS

3.1. *Competitive Paths in the Exhaustible Resource Model*

In the context of the exhaustible resource model, described in Example 2 of Section 2, the competitive conditions (2.5) and (2.6) amount to two familiar rules: (i) Hotelling's Rule on the allocation of an exhaustible resource over time, and (ii) Ramsey's Rule on the allocation of consumption over time.

We can see this as follows. An interior competitive path $(z(t), k(t), p(t), q(t))$ satisfies

$$\begin{aligned} &H(k(t), p(t), q(t)) \\ &\quad = \text{Maximize} \quad q(t)\, w[G(k_1(t), r) - \delta k_1(t) - z_1] + p_1(t)\, z_1 + p_2(t)(-r) \\ &\quad\quad \text{subject to} \quad (z_1, -r, k_1(t), k_2(t)) \in \Lambda, \end{aligned}$$

where we have written r (the exhaustible resource use) for $(-z_2)$.

Using the first-order conditions for an interior maximum, we get

$$q(t)\, w'(c(t))(-1) + p_1(t) = 0 \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{3.1}$$

$$q(t)\, w'(c(t))[G_2(k_1(t), r(t))] - p_2(t) = 0 \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{3.2}$$

Also, by the envelope theorem, we have

$$\begin{aligned} &\partial H(k(t), p(t), q(t))/\partial k_1 \\ &\quad = q(t)\, w'(c(t))[G_1(k_1(t), r(t)) - \delta] \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \end{aligned} \tag{3.3}$$

$$\partial H(k(t), p(t), q(t))/\partial k_2 = 0 \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{3.4}$$

Thus, using condition (2.6) of a competitive path, we have

$$\dot{p}_1(t) = -q(t)\, w'(c(t))[G_1(k_1(t), r(t)) - \delta] \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{3.5}$$

$$\dot{p}_2(t) = 0 \qquad \text{for} \quad t \geqslant 0 \quad a.e. \tag{3.6}$$

We can use (3.1) and (3.5) to study the price path of the augmentable capital good:

$$[\dot{p}_1(t)/p_1(t)] = [G_1(k_1(t), r(t)) - \delta] \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{3.7}$$

The price of the exhaustible resource is related to that of the augmentable good through (3.1) and (3.2):

$$p_2(t) = p_1(t)\, G_2(k_1(t), r(t)) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{3.8}$$

Differentiating (3.8) with respect to t, and using (3.6), we get

$$0 = \dot{p}_2(t) = \dot{p}_1(t)\, G_2(k_1(t), r(t)) + \dot{G}_2(k_1(t), r(t))\, p_1(t).$$

This can be used with (3.7) to get

$$[\dot{G}_2(k_1(t), r(t))/G_2(k_1(t), r(t))]$$
$$= -[\dot{p}_1(t)/p_1(t)] = [G_1(k_1(t), r(t)) - \delta] \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.}, \tag{3.9}$$

which indicates that the rates of return on the two capital goods (the augmentable and the non-renewable one) are equalized. This is known as *Hotelling's rule*.

The version of *Ramsey's rule* that emerges in this context can be derived by differentiating (3.1)

$$\dot{p}_1(t) = \dot{q}(t)\, w'(c(t)) + q(t)\, \dot{w}'(c(t)) \tag{3.10}$$

and then combining (3.1), (3.7), and (3.10) to yield

$$-[\dot{w}'(c(t))/w'(c(t))] = [\dot{q}(t)/q(t)] + [G_1(k_1(t), r(t)) - \delta]. \tag{3.11}$$

(If the utility weights, $q(t)$, happen to be exponential, (3.11) yields the standard Ramsey–Euler equation of optimal growth theory with exponential discounting.)

Given concavity of the production and welfare functions, the Hotelling Rule (3.9) and the Ramsey Rule (3.11) in fact *characterize* competitive paths in this framework.

3.2. *Hartwick's Result in the Exhaustible Resource Model*

Hartwick [14] made the observation that if along a competitive path one invests resource rents in the accumulation of the (augmentable) capital good, that is,

$$\dot{k}_1(t) = z_1(t) = r(t)\, G_2(k_1(t), r(t)) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.}, \tag{3.12}$$

then the path is equitable:

$$c(t)\ [\text{and hence } w(c(t))] \quad \text{is constant over time.} \tag{3.13}$$

We refer to (3.12) as *Hartwick's* (investment) *rule*.

Hartwick's observation may be seen as follows. Using the feasibility condition

$$c(t)=G(k_1(t), r(t))-\delta k_1(t)-z_1(t)$$

and Hartwick's rule (3.12), we get

$$c(t)=G(k_1(t), r(t))-\delta k_1(t)-r(t)\, G_2(k_1(t), r(t)) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{3.14}$$

Differentiating (3.14), one obtains

$$\begin{aligned}\dot{c}(t) &= G_1(k_1(t), r(t))\, \dot{k}_1(t)+G_2(k_1(t), r(t))\, \dot{r}(t)-\delta \dot{k}_1(t)\\ &\quad -r(t)\, \dot{G}_2(k_1(t), r(t))-\dot{r}(t)\, G_2(k_1(t), r(t))\\ &= [G_1(k_1(t), r(t))-\delta]\, \dot{k}_1(t)-r(t)\, \dot{G}_2(k_1(t), r(t))\\ &= [G_1(k_1(t), r(t))-\delta]\, \dot{k}_1(t)\\ &\quad -[\dot{G}_2(k_1(t), r(t))/G_2(k_1(t), r(t))]\, r(t)\, G_2(k_1(t), r(t)).\end{aligned}$$

Using Hartwick's rule (3.12) again, we get

$$\dot{c}(t)=\{[G_1(k_1(t), r(t))-\delta]-[\dot{G}_2(k_1(t), r(t))/G_2(k_1(t), r(t))]\}\, \dot{k}_1(t),$$

which yields $\dot{c}(t)=0$ by using Hotelling's Rule (3.9). This establishes (3.13), that is, intertemporal equity.

3.3. *Dixit, Hammond, and Hoel's Result*

Dixit *et al.* [12] observed that Hartwick's rule could be restated as the condition that the total value of investment in all capital goods is zero:

$$p(t)\, \dot{k}(t)=0. \tag{3.15}$$

In the context of the exhaustible resource model, we have, by using (3.8),

$$p(t)\, \dot{k}(t)=p_1(t)\, \dot{k}_1(t)+p_2(t)(-r(t))=p_1(t)[\dot{k}_1(t)-r(t)\, G_2(k_1(t), r(t))],$$

so that (3.15) is equivalent to (3.12).

The question arises whether Hartwick's rule, in the form (3.15), for general capital accumulation models with heterogenous capital goods, *characterizes* intertemporal equity along competitive paths. The answer is that a weaker condition than (3.15) actually provides such a characterization. This weaker condition, which can be called the *Dixit–Hammond–Hoel rule,* is that the value of investment along a competitive path, $p(t)\, z(t)$, is constant over time.

We formally state and prove this result below.

PROPOSITION 1. *An interior competitive path* $(z(t), k(t), p(t), q(t))$ *from* K *in* $\mathbb{R}^n_{++}$ *is equitable iff*

$$I(t) \equiv p(t)\, z(t) \qquad \text{is constant over time.} \tag{3.16}$$

Proof. Since the competitive path is interior, we have from Lemma 1(ii)

$$\dot{y}(t) = \dot{q}(t)\, u(z(t), k(t)) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{3.17}$$

Differentiating (2.8), we get

$$\dot{y}(t) = q(t)\, \dot{u}(z(t), k(t)) + \dot{q}(t)\, u(z(t), k(t)) + \dot{I}(t) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{3.18}$$

Using (3.17) and (3.18) we obtain

$$\dot{I}(t) = -q(t)\, \dot{u}(z(t), k(t)) \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{3.19}$$

Clearly (3.19) yields the equivalence of $\dot{u}(z(t), k(t)) = 0$ and $\dot{I}(t) = 0$ (since $q(t) > 0$). ∎

3.4. *An Example*

It should not be concluded from Proposition 1 above, that Hartwick's rule (3.15) does not characterize equitable competitive paths. It is logically possible that if a competitive path satisfies the Dixit–Hammond–Hoel rule (3.16), it necessarily satisfies Hartwick's rule (3.15). That is, whenever the value of net investment is constant over time along a competitive path, the constant is zero. Indeed, for a class of exhaustible resource models of the type described in Example 2 of Section 2, this is precisely what happens (see Dasgupta and Mitra [11] for this intriguing result).

We now indicate, in an example, that Hartwick's rule can be violated for an interior competitive path which is equitable, so that Hartwick's rule is, in general, different from the Dixit–Hammond–Hoel rule, and therefore it does *not* characterize equitable competitive paths.

Our discussion will be based on Example 1 of Section 2, the familiar one-sector neoclassical model of growth. The function $[G(k) - \delta(k)]$ is maximized at a unique point, k^*, among all $k \in [0, \infty)$, and $0 < k^* < \infty$. By the assumptions on G and δ, $[G(k^*) - \delta k^*] > 0$ and is denoted by C^*. Then (k^*, C^*) denotes the *golden-rule* capital and consumption levels, respectively. There is a unique point, $\bar{k}$, in $(0, \infty)$ where $G(\bar{k}) = \delta \bar{k}$. One can check that $k^* < \bar{k} < \infty$; $\bar{k}$ is the *maximum sustainable stock*.

Let the initial capital stock, K, be in $(k^*, \bar{k})$. Consider the differential equation

$$\dot{k}(t) = G(k(t)) - \delta k(t) - C^*, \qquad k(0) = K \in (k^*, \bar{k}). \tag{3.20}$$

It can be checked that there is a unique solution $(k(t))$ to this differential equation, and $(k(t))$ has the following properties:

$$\begin{aligned} &\text{(i)} \quad k^* < k(t) < K && \text{for} \quad t \geqslant 0; \\ &\text{(ii)} \quad -\delta k(t) < \dot{k}(t) < 0 && \text{for} \quad t \geqslant 0 \end{aligned} \tag{3.21}$$

Defining $z(t) = \dot{k}(t)$ for $t \geqslant 0$, we see that $(z(t), k(t))$ is an interior path from K, and $u(z(t), k(t)) = w(C^*)$ for $t \geqslant 0$. Thus, $(z(t), k(t))$ is an equitable path.

Let us define a function $p(t)$ by

$$p(t) = \exp\left[\int_0^t (\delta - G'(k(s)))\, ds\right]. \tag{3.22}$$

Then $p(0) = 1$ and $p(t)$ is increasing with t. Denoting $[p(t)/w'(C^*)]$ by $q(t)$, it is straightforward to check that $(z(t), k(t), p(t), q(t))$ is an interior competitive path from K.

Since $\dot{k}(t) < 0$, it is clear that

$$p(t)\, \dot{k}(t) < 0 \qquad \text{for} \quad t \geqslant 0, \tag{3.23}$$

so Hartwick's rule is violated on this competitive equitable path.

The competitive equitable path which we have constructed is clearly inefficient To see this, define T by

$$(k^*/K) = 1/e^{\delta T}. \tag{3.24}$$

Then define the function $k'(t)$ for $0 \leqslant t \leqslant T$ by

$$k'(t) = K/e^{\delta t} \tag{3.25}$$

and $k'(t) = k^*$ for $t > T$. Then denoting $\dot{k}'(t)$ by $z'(t)$ for all $t \neq T$, $(z'(t), k'(t))$ is a path from K, which satisfies

$$G(k'(t)) - \delta k'(t) - \dot{k}'(t) > C^* \qquad \text{for} \quad 0 \leqslant t < T$$

and $G(k'(t)) - \delta k'(t) - \dot{k}'(t) = C^*$ for $t > T$. Thus, $(z(t), k(t))$ could not be efficient.

4. A CHARACTERIZATION OF EFFICIENT EQUITABLE PATHS

Since Hartwick's rule does not characterize competitive equitable paths, we are led to evaluate the precise significance of Hartwick's rule. The example of Section 3.4 seems to point in the following direction: if along a competitive equitable path, Hartwick's rule is *not* satisfied, then the path is intertemporally *inefficient*. If this were true in general, then Hartwick's rule would take on a significance that is quite distinct from the intertemporal equity issue. In fact, its role then would be to help signal inefficiency of competitive equitable paths. Along any competitive equitable path, the value of net investment would be constant; but, if the constant were non-zero, the path would be pronounced inefficient.

A word of caution: we are *not* saying that Hartwick's rule will identify *efficient* paths. It is possible for a competitive equitable path to satisfy HR and still be inefficient. One only needs to consider a stationary path in the neoclassical one-sector model, with the stationary capital stock exceeding the golden-rule capital stock.

We proceed, in this section, to show that the lesson of the example of Section 3.4 can be fully generalized. But such a demonstration clearly requires a convenient necessary condition of efficient paths, which would yield Hartwick's rule when the path was also equitable. Absent any such general necessary condition in continuous time models, we have to develop one. This is accomplished in Theorem 1. The result on intertemporal efficiency is then used in Theorem 2 to derive Hartwick's rule as a necessary condition for efficient equitable paths.

4.1. *Efficiency and Terminal Cost Minimization*

The basic necessary condition for competitive efficiency that we establish is "terminal cost minimization," a concept due to Malinvaud [18]. In fact, in his general discrete-time model, Malinvaud characterized competitive efficiency in terms of this condition.

The concept of terminal cost minimization may be explained as follows. Consider a competitive efficient path $(z(t), k(t), p(t), q(t))$, and consider any instant of time, $T > 0$. Then for any path $(z'(t), k'(t))$ which maintains the same utility stream as $(z(t), k(t))$ from T onwards (that is, $u(z'(t), k'(t)) = u(z(t), k(t))$ for $t \geqslant T$ a.e.), we must have the "terminal cost" of the capital stocks at T on the path $(z'(t), k'(t))$ at least as high as on the path $(z(t), k(t))$ (that is, $p(T)\, k'(T) \geqslant p(T)\, k(T)$). And this property is true for an arbitrary $T > 0$.

THEOREM 1. *Let $(z(t), k(t), p(t), q(t))$ be an interior competitive path from $\bar{K}$, which is efficient. Then for a.e. $T > 0$,*

$$p(T)\, k(T) \leqslant p(T)\, k \tag{4.1}$$

for all $k \in A \equiv \{K$: *there is a path* $(\hat{z}(t), \hat{k}(t))$ *from* K *satisfying* $u(\hat{z}(t), \hat{k}(t)) \geqslant u(z(t+T), k(t+T))$ *for a.e.* $t \geqslant 0\}$.

Proof. Pick any $T > 0$. Suppose, contrary to (4.1), there is $\bar{k} \in A$, such that

$$p(T)\,k(T) > p(T)\,\bar{k}. \tag{4.2}$$

We will show that this supposition leads to a contradiction. To this end, denote $[k(T) - \bar{k}]$ by h; then $p(T)\,h > 0$.

Step 1. Denote $\|u_1(z(T), k(T))\|$ by B'; let M' be the maximum value of $-[V'F(z(T), k(T))\,V]$ among all $V = (a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying $\|V\| \leqslant 1$, where $F(z(T), k(T))$ is the Hessian of u, evaluated at $(z(T), k(T))$. Let $(B, M) \gg (B', M')$.

Pick $\theta > 0$, such that for all (k, z) satisfying $\|z - z(T), k - k(T)\| \leqslant \theta$, we have $(z, k) \in$ *int* A, and

(i) $-[V'F(z, k)\,V] \leqslant 2M$ for all $V = (a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying $\|V\| \leqslant 1$;

(ii) $\|u_1(z, k)\| \leqslant 2B$.

By definition of (B, M), this can be done.

Step 2. Using the continuity of $(z(t), k(t), p(t), q(t))$ at $t = T$, we can find $0 < S < T$, such that for all $t \in [S, T]$,

(i) $\|z(t) - z(T), k(t) - k(T)\| \leqslant (\theta/4)$

(ii) $\|u_2(z(t), k(t))\|\,\|h\|(T - S) < (1/4)[p(T)\,h/q(T)]$

(iii) $[p(t)\,h/q(t)] \geqslant (3/4)[p(T)\,h/q(T)]$.

Step 3. Choose $0 < \lambda < 1$, with λ sufficiently close to zero, so that

$$\begin{aligned} &\text{(i)} \quad [\lambda^{1/4}\,\|h\|/(T-S)][1 + (T-S)] \leqslant (\theta/4); \\ &\text{(ii)} \quad 2\theta^2\lambda^{1/2}M < p(T)\,hq(T)/(T-S). \end{aligned} \tag{4.3}$$

Define the function $(z'(t), k'(t))$ from $\bar{K}$ by

$$\begin{aligned} z'(t) &= z(t) && \text{for} \quad 0 \leqslant t < S \\ z'(t) &= z(t) - [\lambda h/(T-S)] && \text{for} \quad t \in [S, T] \\ z'(t) &= (1-\lambda)\,z(t) + \lambda\hat{z}(t-T) && \text{for} \quad t > T \end{aligned}$$

and:

$$k'(t) = \bar{K} + \int_0^t z'(s)\, ds \qquad \text{for} \quad t \geqslant 0$$

We can check that $(z'(t), k'(t))$ is a path from $\bar{K}$. First, note that $k'(t) = k(t)$ for $0 \leqslant t \leqslant S$. For $t \in [S, T]$, we have

$$k'(t) = \bar{K} + \int_0^t z'(s)\, ds = -\lambda \int_S^t [\lambda h/(T-S)]\, ds + \bar{K} + \int_0^t z(s)\, ds$$
$$= -[\lambda h(t-S)/(T-S)] + k(t).$$

Thus, for $t \in [S, T]$, we have $\|z'(t) - z(t), k'(t) - k(t)\| = [\lambda \|h\|/(T-S)] \times [1 + (t-S)] \leqslant \lambda^{3/4}(\theta/4) \leqslant (\theta/4)$ by (4.3), so that by using Step 2(i), we have $\|z'(t) - z(T), k'(t) - k(T)\| \leqslant \theta$ for $t \in [S, T]$. Consequently, $(z'(t), k'(t)) \in int\, \Lambda$ by Step 1. Further $k'(T) = k(T) - \lambda h = k(T) - \lambda[k(T) - \bar{k}] = (1-\lambda)\, k(T) + \lambda \bar{k}$. Since $\hat{k}(0) = \bar{k}$, we have, for $t > T$, $(z'(t), k'(t)) = (1-\lambda)(z(t), k(t)) + \lambda(\hat{z}(t-T), \hat{k}(t-T))$, and consequently $(z'(t), k'(t)) \in \Lambda$ by convexity of Λ.

Step 4. Clearly, for $0 \leqslant t < S$, we have $u(z'(t), k'(t)) = u(z(t), k(t))$. And, for $t > T$, we have $u(z'(t), k'(t)) \geqslant (1-\lambda)\, u(z(t), k(t)) + \lambda u(\hat{z}(t-T), \hat{k}(t-T)) \geqslant u(z(t), k(t))$, by definition of A.

We will now show that for $t \in [S, T]$, we have

$$u(z'(t), k'(t)) > u(z(t), k(t)). \tag{4.4}$$

For $t \in [S, T]$, we calculate

$$\begin{aligned} &u'(z'(t), k'(t)) - u(z(t), k(t)) \\ &\quad = u_1(z(t), k(t))(z'(t) - z(t)) + u_2(z(t), k(t))(k'(t) - k(t)) \\ &\qquad + (1/2)\, V'(t)\, F(\tilde{z}(t), \tilde{k}(t))\, V(t), \end{aligned} \tag{4.5}$$

where $V(t) = (z'(t) - z(t), k'(t) - k(t))$ and $(\tilde{z}(t), \tilde{k}(t))$ is a convex combination of $(z(t), k(t))$ and $(z'(t), k'(t))$ as given by Taylor's expansion.

Note that since $\|z'(t) - z(T), k'(t) - k(T)\| \leqslant \theta$ (by Step 3) and $\|z(t) - z(T), k(t) - k(T)\| \leqslant \theta$ (by Step 2), we have $\|\tilde{z}(t) - z(T), \tilde{k}(t) - k(T)\| \leqslant \theta$. Thus $(\tilde{z}(t), \tilde{k}(t)) \in int\, \Lambda$, and $-[V'F(\tilde{z}(t), \tilde{k}(t))\, V] \leqslant 2M$ for all $V = (a, b)$ in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $\|V\| \leqslant 1$.

Now $\|V(t)\| \leqslant \lambda \|h\|[1 + (T-S)]/(T-S) \leqslant \lambda^{3/4}(\theta/4)$ for $t \in [S, T]$, so that $\|V(t)/\theta\lambda^{3/4}\| \leqslant (1/4) < 1$, and

$$-[V'(t)\, F(\tilde{z}(t), \tilde{k}(t))\, V(t)] \leqslant (\theta^2 \lambda^{3/2})\, 2M. \tag{4.6}$$

Also, $\|k'(t)-k(t)\| \leqslant \lambda \|h\|$ by Step 3, so

$$|u_2(z(t), k(t))(k'(t)-k(t))| \leqslant \|u_2(z(t), k(t)\| \, \|k'(t)-k(t)\| < (\lambda/4)[p(T)\, h/q(T)]/(T-S)$$

by Step 2(ii). Finally,

$$\begin{aligned} u_1(z(t), k(t))(z'(t)-z(t)) &= -[p(t)(z'(t)-z(t))/q(t)] \\ &= [p(t)/q(t)][\lambda h/(T-S)] \\ &\geqslant (3\lambda/4)[p(T)\, h/q(T)]/(T-S) \end{aligned}$$

by Step 2(iii). Thus, we have

$$\begin{aligned} &u_1(z(t), k(t))(z'(t)-z(t)) + u_2(z(t), k(t))(k'(t)-k(t)) \\ &\quad \geqslant (\lambda/2)[p(T)\, h/q(T)]/(T-S). \end{aligned} \tag{4.7}$$

Using (4.6) and (4.7) in (4.5), we obtain

$$\begin{aligned} u(z'(t), k'(t)) - u(z(t), k(t)) &\geqslant (\lambda/2)\{[p(T)\, h/q(T)]/(T-S)\} - \theta^2\lambda^{3/2}M \\ &= (\lambda/2)\{[p(T)\, h/q(T)(T-S)] - \theta^2\lambda^{1/2}2M\} \\ &> 0 \end{aligned}$$

by (4.3)(ii). This proves (4.4), and establishes that $(z(t), k(t))$ is inefficient, a contradiction. Thus (4.1) must hold. ∎

4.2. *Efficient Equitable Paths Satisfy Hartwick's Rule*

We are now in a position to establish the principal result of this paper: efficient equitable paths must satisfy Hartwick's investment rule. The reason that the terminal cost minimization property of competitive efficient paths is just the right concept is establishing this result is the following. Given a competitive efficient path $(z(t), k(t), p(t), q(t))$ which also happens to be equitable, it can be compared with paths which are the same as the path $(z(t), k(t), p(t), q(t))$, except that they are shifted forward in time or backward in time (by any small fixed time interval). Terminal cost minimization will then imply that for each $T > 0$, $p(T)\, k(S)$ is *minimized* at T among all S in a neighborhood of T. Then Hartwick's rule,

$$p(T)\, \dot{k}(T) = 0,$$

is simply the necessary first-order condition of such a minimum.[11]

[11] Note that our earlier "word of caution" translates to the following observation. Since the terminal cost-minimization condition characterizes competitive efficiency, and Hartwick's rule is only the first-order necessary condition of this minimization, it cannot *identify* competitive efficiency.

THEOREM 2. *Let $(z(t), k(t), p(t), q(t))$ be an interior competitive path, which is efficient and equitable. Then*

$$p(t)\, z(t) = 0 \qquad \text{for} \quad t \geqslant 0, \quad \text{a.e.} \tag{4.8}$$

Proof. Pick any $T > 0$ at which k is differentiable (as a function of t) and at which z is continuous (as a function of t). The complement of this set of T in $[0, \infty)$ has Lebesgue measure zero.

Pick $S > T$, and note that k_S is in A, where A is defined in Theorem 1, since the given path is equitable. Thus, using Theorem 1, we have

$$p(T)(k(T) - k(S)) \leqslant 0.$$

This implies that, since $S > T$,

$$p(T)[k(S) - k(T)]/(S - T) \geqslant 0. \tag{4.9}$$

Letting $S \to T$, and noting that k is differentiable at T, we have

$$p(T)\, z(T) = p(T)\, \dot{k}(T) \geqslant 0. \tag{4.10}$$

Pick $0 < s < T$ and note that k_s is in A, where A is defined in Theorem 1, since the given path is equitable. Thus, using Theorem 1, we have

$$p(T)(k(T) - k(s)) \leqslant 0.$$

This implies that since $s < T$,

$$p(T)[k(T) - k(s)]/(T - s) \leqslant 0. \tag{4.11}$$

Letting $s \to T$, and noting that k is differentiable at T, we have

$$p(T)\, z(T) = p(T)\, \dot{k}(T) \leqslant 0. \tag{4.12}$$

Combining (4.10) and (4.12), we get (4.8). ▮

REFERENCES

1. E. Burmeister and P. J. Hammond, Maximin paths of heterogeneous capital accumulation and the instability of paradoxical steady states, *Econometrica* **45** (1977), 853–870.
2. E. Burmeister and S. J. Turnovsky, Capital deepening response in an economy with heterogeneous capital goods, *Amer. Econ. Rev.* **62** (1972), 842–853.
3. D. Cass, Optimum growth in an aggregative model of capital accumulation, *Rev. Econ. Stud.* **32** (1965), 233–240.
4. D. Cass, On capital over-accumulation in the aggregative neoclassical model of economic growth: A complete characterization, *J. Econ. Theory* **4** (1972), 200–223.

5. D. Cass and K. Shell, The structure and stability of competitive dynamical systems, *J. Econ. Theory* **12** (1976), 31–70.
6. P. S. Dasgupta and G. M. Heal, The optimal depletion of exhaustible resources, *in* "Symposium on the Economics of Exhaustible Resources," *Rev. Econ. Stud.* **41** (1974), 3–28.
7. P. S. Dasgupta and G. M. Heal, "Economic Theory and Exhaustible Resources," Cambridge Univ. Press, Cambridge, UK, 1979.
8. S. Dasgupta and T. Mitra, Intergenerational equity and efficient allocation of exhaustible resources, *Int. Econ. Rev.* **24** (1983), 133–153.
9. S. Dasgupta and T. Mitra, "On the Welfare Significance of National Product for Economic Growth and Sustainable Development," CAE Working Paper 98-09, Cornell University, 1998.
10. S. Dasgupta and T. Mitra, On the welfare significance of national product for economic growth and sustainable development, *Jpn. Econ. Rev.* **50** (1999), 422–442.
11. S. Dasgupta and T. Mitra, Intertemporal equity and Hartwick's rule in an exhaustible resource model, mimeo, Cornell University, 2002.
12. A. Dixit, P. Hammond, and M. Hoel, On Hartwick's rule for regular maximin paths of capital accumulation and resource depletion, *Rev. Econ. Stud.* **47** (1980), 551–556.
13. R. Dorfman, P. A. Samuelson, and R. M. Solow, "Linear Programming and Economic Analysis," McGraw–Hill, New York, 1958.
14. J. M. Hartwick, Intergenerational equity and investing of rents from exhaustible resources, *Amer. Econ. Rev.* **66** (1977), 972–974.
15. T. C. Koopmans, On the concept of optimal economic growth, *Pontif. Acad. Sci.* (1965), 225–288.
16. M. Magill, Infinite horizon programs, *Econometrica* **49** (1981), 679–711.
17. M. Majumdar, Some general theorems on efficiency prices with an infinite dimensional commodity space, *J. Econ. Theory* **5** (1972), 1–13.
18. E. Malinvaud, Capital accumulation and efficient allocation of resources, *Econometrica* **21** (1953), 233–268.
19. R. M. Solow, Intergenerational equity and exhaustible resources, *in* "Symposium on the Economics of Exhaustible Resources," *Rev. Econ. Stud.* **41** (1974), 29–45.
20. R. M. Solow, On the intergenerational allocation of natural resources, *Scand. J. Econ.* **88** (1986), 141–149.
21. R. M. Solow, "An Almost Practical Step Towards Sustainability," Resources for the Future, Washington, D.C., 1992.
22. S. I. Takekuma, A support price theorem for the continuous time model of capital accumulation, *Econometrica* **50** (1982), 427–442.
23. M. L. Weitzman, On the welfare significance of national product in a dynamic economy, *Quart. J. Econ.* **90** (1976), 156–162.
24. C. Withagen and G. M. Asheim, Characterizing sustainability: The converse of Hartwick's rule, *J. Econ. Dynam. Control* **23** (1998), 159–165.